(Defn) **Linear transformation**:- Let E and F be two linear spaces over a field K. A map $T: E \to F$ is called a linear transformation if for every $x, y \in E$ & $\alpha \in K$,

$$T(x+y) = T(x) + T(y).$$

$$\& \; T(\alpha x) = \alpha T(x).$$

Example:- Let $T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 2x$ for all $x \in \mathbb{R}$.

Then, $T(x+y) = 2(x+y) = 2x + 2y = T(x) + T($ for all $x, y \in \mathbb{R}$. and for any $\alpha \in K$, $T(\alpha, x) = 2$ $= \alpha \cdot 2x = \alpha \cdot T(x)$ for all $x \in \mathbb{R}$.

$\therefore T$ is a linear transformation $T(x) = x^2$, then T is not a linear transformation.

Continuous linear transformation:- Let E and F be normed linear spaces. A linear transformation T from E into F is said to be continuous if T is continuous as a map from the metric space E to the metric space F.

i.e. if for every $x \in E$ and every sequence $\{x_n\}$ in E converging to x the sequence $\{T(x_n)\}$ converges to $T(x)$. i.e. $\|x_n - x\| \to 0$ as $\|T(x_n) - T(x)\| \to 0$ as $n \to \infty$.

Theorem (1):> If T is a Continuous linear transformation from a nls E into a nls F then ker(T) is a closed linear subspace of E.

Proof:- We have $\ker T = \{x \in E : T(x) = 0\}$. Let $x_1, x_2 \in \ker T$ and $\alpha_1, \alpha_2 \in K$, then

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$$
$$= \alpha_1 \cdot 0 + \alpha_2 \cdot 0 = 0$$

$\therefore \alpha_1 x_1 + \alpha_2 x_2 \in \ker T$.

So ker T is a linear subspace of T.

Now, $\{0\}$ is closed in F. Since T is Continuous $T^{-1}\{0\} = \ker T$ is a closed set in E.

Theorem (2):> A linear transformation T from a nls E to a nls F is Continuous iff it is Continuous at the origin.

Proof:- Suppose T is Continuous. Then it is Continuous at every Point of E. Hence T is Continuous at the origin.

Conversely, let T be Continuous at the origin then, $\because T(0) = 0$ given $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - 0\| < \delta \Rightarrow \|T(x) - T(0)\| < \epsilon$

i.e. $\|x\| < \delta \Rightarrow \|T(x)\| < \epsilon$.

Now, for all $x, y \in E$, $\|x - y\| < \delta \Rightarrow \|T(x - y)\| = \|T(x) - T(y)\| < \epsilon$.

$\therefore$ T is uniformly Continuous.

Hence T is Continuous.

Q.No → Show that a non-empty subset of a normed linear space N is bounded if and only if f[S].

~~or, f~~

Proof:- Let S be bounded in N. Therefore there exist a +ve real number α such that $\|x\| \leq \alpha$ for every $x \in S$. If $f \in N^*$

$\therefore |f(x)| \leq \|f\| \|x\| \leq \alpha \|f\|$ for every $x \in S$ so $f(S)$ is a bounded set of numbers.

Conversely, we suppose $f(S)$ is bounded for each $f \in N^*$. For each fixed $x \in S$, we define a linear functional F_x on the dual space N^* by seting $F_x(f) = f(x)$ for each $f \in N^*$, it is easiy to verify that F_x is linear.

Now, $\|F_x\| = \sup\limits_{\|f\|=1} |F_x(f)| = \sup\limits_{\|f\|=1} |f(x)| = \|x\|$ — ①

Hence, F_x is a continuous linear functional on the Banach space N^*, By the hypothesis for each $f \in N^*$,

$$\sup_{x \in S} |F_x(f)| = \sup_{x \in S} |f(x)| < \infty.$$

Therefore, by ① the Principle of uniform bounded,

$$\sup_{x \in S} \|x\| = \sup_{x \in S} \|F_x\| < \infty.$$

Hence, S is bounded.